Exercises for the 6th SIPTA Summer School

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July 21-25, 2014 – Montpellier, France

1 Presentation examples

1. Are the following bets rational?

   - $$P(\{C\}) = 0.5, \quad P(\{C,P\}) = 0.3$$

     No, I sell one ticket of the first and buy one of the second, then my gain is

     $$(0.5 - \mathbb{I}_C) + (\mathbb{I}_{P,C} - 0.3) = 0.2 + \mathbb{I}_{P,C} - \mathbb{I}_C$$

     and the second part is always positive or null

   - $$P(\{C,P\}) = 0.5, \quad P(\{P,T\}) = 0.6, \quad P(\{C,T\}) = 0.7$$

     No, I buy each one of the ticket, then my gain is

     $$\mathbb{I}_{P,C} + \mathbb{I}_{P,T} + \mathbb{I}_{C,T} - 1.8 = 0.2$$

   - $$P(\{C,P\}) = 0.5, \quad P(\{P,T\}) = 0.5$$

     Yes, since
     - If I only buy tickets, more of the first than the second, I lose if T happens
     - If I only sell tickets, I lose if P happens
     - If I buy and sell, buying more of the first than selling the second, I lose if T happens

     Also, the probability $p(C) = p(T) = 0.5$ and $p(P) = 0$ represents the assessments

2. Are the following bets rational?

   (a) – $X_1(R) = 20$, $X_1(P) = -40$, $X_1(D) = 60$, $\underline{P}(X_1) = 20$
       – $X_2(R) = -20$, $X_2(P) = 40$, $X_2(D) = -60$, $\underline{P}(X_2) = -10$

   No, simply sum the two bets to obtain no reward and a negative value

   (b) – $X_1(R) = 10$, $X_1(P) = -20$, $X_1(D) = 0$, $\underline{P}(X_1) = 0$
       – $X_2(R) = 0$, $X_2(P) = 10$, $X_2(D) = 0$, $\underline{P}(X_2) = 5$

   No. Assume I sell the first and 2.1 of the second, then
   - if R happens, my reward (and your loss) is 10.5 - 10=0.5
   - if P happens, my reward (and your loss) is 10.5 +20 -10=20.5
   - if D happens, my reward (and your loss) is 10.5

   (c) – $X_1(R) = 0$, $X_1(P) = 20$, $X_1(D) = -10$, $\underline{P}(X_1) = 0$
       – $X_2(R) = 20$, $X_2(P) = 0$, $X_2(D) = -10$, $\underline{P}(X_2) = 0$

   Yes
   - If I only sell or buy, then there is a winning and a losing situation, since signs for R, P and D are opposite
   - If I buy and sell, then R and P will be losing and winning situations

3. Are the following bets rational?

   (a) – $X_1(C) = 1$, $X_1(P) = -1$, $X_1(T) = 0$, $P(X_1) = 0$
       – $X_2(C) = 0$, $X_2(P) = 1$, $X_2(T) = -1$, $P(X_2) = 0$
       – $X_3(C) = 1$, $X_3(P) = -1$, $X_3(T) = 1$, $P(X_3) = 0$

   No, assume you sum (buy) the first and subtract (sell) 1.2 the second and 1.5 the third, then your gain is
   - if C, $X_1(C) - 1.2X_2(C) - 1.5X_3(C) = -0.5$

– if P, $X_1(P) - 1.2X_2(P) - 1.5X_3(P) = -0.7$
– if T, $X_1(T) - 1.2X_2(T) - 1.5X_3(T) = -0.3$

(b) – $X_1(C) = 2$, $X_1(P) = -1$, $X_1(T) = 0$, $P(X_1) = 1$
– $X_2(C) = 1$, $X_2(P) = 1$, $X_2(T) = -1$, $P(X_2) = 0$

Yes, because buying/seeling λ_1 units of X_1 gives you the rewards $\lambda_1(1, -2, 0)$ (with $\lambda_1 > 0$ if you buy and $\lambda_1 < 0$ if you sell). Similarly, buying/seeling λ_2 units of X_2 gives you the rewards $\lambda_2(1, 1, -1)$ (with $\lambda_2 > 0$ if you buy and $\lambda_2 < 0$ if you sell). Then:

– If you only buy ($\lambda_1, \lambda_2 \geq 0$, C will make you gain.
– if you only sell ($\lambda_1, \lambda_2 \leq 0$, T will make you gain.
– If you sell X_1 ($\lambda_1 < 0$) and buy X_2 ($\lambda_2 > 0$), P will make you gain.
– If you sell X_2 ($\lambda_2 < 0$) and buy X_1 ($\lambda_1 > 0$), either C or T will make you gain (they give you opposite rewards).

Also, the probability $p(T) = p(C) = 0.5$ and $p(P) = 0$ gives these expectations.

2 Exercises

1. Form group of 2/3 persons, each person assessing separately price for **one (or two) different** gamble among the three following ones that give 1 if events/assertions are true: Montpellier agglomeration (city + neighbouring villages) counts between

   - 300K and 400K inhabitants
   - 400K and 500K inhabitants
   - 500K and 600K inhabitants

   Compare your assessments. Discuss how you did to reach such prices? Are they rational? Can you agree on rational prices?

2. Represent the assessments

   - $X_1(C) = 1$, $X_1(P) = -1$, $X_1(T) = 0$, $P(X_1) = 0$
   - $X_2(C) = 0$, $X_2(P) = 1$, $X_2(T) = -1$, $P(X_2) = 0$
   - $X_3(C) = 1$, $X_3(P) = -1$, $X_3(T) = 1$, $P(X_3) = 0$

   and

   - $X_1(C) = 2$, $X_1(P) = -1$, $X_1(T) = 0$, $P(X_1) = 1$
   - $X_2(C) = 1$, $X_2(P) = 1$, $X_2(T) = -1$, $P(X_2) = 0$

   In a geometric way, and check in this way that they are rational or not

3. John is planning to bet on the winner of the Formula 1 championship.

(a) Determine the set of probabilities compatible with his beliefs, if he thinks that:

- Only one of Rosberg, Hamilton, Alonso or Vettel can win.
- The probability of Hamilton winning is at least twice as much of that of Alonso winning, and this is at least 1.5 times the probability of Vettel winning.
- Rosberg has exactly the same probability of winning than Hamilton.

The above assessments mean that:

$$\begin{aligned} &P(\text{Rosberg}) + P(\text{Hamilton}) + P(\text{Alonso}) + P(\text{Vettel}) = 1 \\ &P(\text{Hamilton}) \geq 2P(\text{Alonso}) \\ &P(\text{Alonso}) \geq 1.5P(\text{Vettel}) \\ &P(\text{Rosberg}) = P(\text{Hamilton}). \end{aligned}$$

The set of probabilities compatible with these assessments has extreme points

$$\left\{(0.5, 0.5, 0, 0) \ \ (\frac{3}{7}, \frac{3}{7}, \frac{1}{7}, 0), \ \ (\frac{6}{17}, \frac{6}{17}, \frac{3}{17}, \frac{2}{17})\right\},$$

where the denote $(a, b, c, d) = (P(\text{Rosberg}), P(\text{Hamilton}), P(\text{Alonso}), P(\text{Vettel}))$.

(b) They offer him a bet with reward 10 if Alonso wins, 5 if Vettel wins, and -3 if either Rosberg or Hamilton win. According to his beliefs, which are the minimum and maximum expected gains? Should he accept this bet or not?

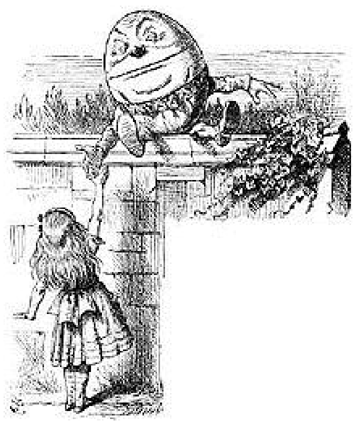
Using the above extreme points, we deduce that the lower and upper previsions of this gamble are

$$\underline{P}(X) = -3 \ \ \overline{P}(X) = \frac{4}{17}.$$

Since the lower prevision of this gamble is negative, it should not be desirable to our subject.

4. Before jumping off the wall, Humpty Dumpty tells Alice the following:

"I have a farm with pigs, cows and hens. There are at least as many pigs as cows and hens together and at least as many hens as cows. How many pigs, cows and hens do I have?"



(a) Determine the set of probabilities compatible with these assessments.

Any probability measure P compatible with these assessments should satisfy $P(\text{pigs}) \geq P(\text{hens}) + P(\{cows\})$, and $P(\text{hens}) \geq P(\text{cows})$. The extreme points of the set of compatible probabilities are

$$\left\{(1, 0, 0), (0.5, 0.5, 0), (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})\right\},$$

where any vector (p_1, p_2, p_3) above denotes $(P(\text{pigs}), P(\text{hens}), P(\text{cows}))$.

(b) Can we express them using lower and upper probabilities of events?

The assessments above mean that $\underline{P}(I_{pigs} - I_{hens,cows}) \geq 0$ and $\underline{P}(I_{hens} - I_{cows}) \geq 0$. While the first is equivalent to $\underline{P}(\text{pigs}) \geq 0.5$, the second cannot be expressed in terms of constraints on lower probabilities of events. The bounds (lower/upper probabilities) are $(P(\text{pigs}) \in [0.5, 1]$, $P(\text{hens}) \in [0, 0.5], P(\text{cows})) \in [0, 0.25]$ It is easy to see that those constraints define a larger set than the initial ones, as $P(C) = 0.25, P(P) = 0.75$ satisfies them, but not the initial assessments.

Before going away, Humpty Dumpty tells Alice the following: “I see you have some money with you. If you give me a coin, we will wait till an animal come out of the barn, and:

– I will give you 1 if a pig comes out
– I will give you 4 if a hen comes out
– you will give me 1 if a cow comes out

” Will Alice accept the gamble? Would she have accepted if she retained only the lower/upper probabilities?

This can be modelled by the gamble $X(pig) = 1, X(hen) = 4$ and $X(cow) = -1$. With the initial information, we have $\underline{P}(X) = 1.25$ (obtained with the extreme point $(0.5, 0.25, 0.25)$), hence the price is acceptable to Alice. If we retain only the lower/upper probabilities, we have $\underline{P}(X) = 0.5$ (with the extreme point $(0.75, 0, 0.25)$), and Alice will not find acceptable the interesting proposal (since she should give a coin for that, so effectively the prices reduce in 1).

5. Consider an urn with 10 balls, of which 3 are red, and the other 7 are either blue or yellow.

(a) Determine the set $\mathcal{M}$ of linear previsions that represent the possible compositions of the urn.

The set of possible compositions of the urn is given by the table below left. It produces the set of linear previsions given in the table below right (we give the probability mass functions that are their restrictions to events).

Red	Blue	Yellow
3	0	7
3	1	6
3	2	5
3	3	4
3	4	3
3	5	2
3	6	1
3	7	0

P_i	Red	Blue	Yellow
P_1	$3/10$	0	$7/10$
P_2	$3/10$	$1/10$	$6/10$
P_3	$3/10$	$2/10$	$5/10$
P_4	$3/10$	$3/10$	$4/10$
P_5	$3/10$	$4/10$	$3/10$
P_6	$3/10$	$5/10$	$2/10$
P_7	$3/10$	$6/10$	$1/10$
P_8	$3/10$	$7/10$	0

(b) Which are the extreme points of this set?

The extreme points of the *convex hull* of the set of compatible probabilities are P_1, P_8.

(c) What are the lower and upper probabilities that the ball selected is blue?

According to the first item, we obtain $\underline{P}(\text{blue}) = 0, \overline{P}(\text{blue}) = \frac{7}{10}$.

(d) Let X be a gamble given by $X(\text{blue}) = 50, X(\text{red}) = 100, X(\text{yellow}) = 300$. What is the lower prevision of X?

The lower prevision of X is the lower envelope of the set $\{P_1(X), P_2(X), \dots, P_8(X)\}$, which in this case is equal to $\underline{P}(X) = 0.3 \cdot 100 + 0.7 \cdot 50 = 65$.

(e) Do the same for an arbitrary gamble Y.

Again, we have $\underline{P}(Y) = \min\{P_1(Y), P_2(Y), \dots, P_8(Y)\}$; since $P_2, \dots, P_7$ are convex combinations of P_1, P_8, it follows that

$$\begin{aligned}\underline{P}(Y) &= \min\{P_1(Y), P_8(Y)\}\\ &= \min\{0.3Y(\text{blue}) + 0.7Y(\text{yellow}), 0.3Y(\text{blue}) + 0.7Y(\text{red})\}\\ &= 0.3Y(\text{blue}) + 0.7\min\{Y(\text{yellow}), Y(\text{red})\}.\end{aligned}$$

6. Consider $\Omega = \{1, 2, 3\}$, and the gambles and lower previsions given by

$$\begin{aligned}X_1(1) = 1, \quad X_1(2) = 2, \quad X_1(3) = 3\\ X_2(1) = 3, \quad X_2(2) = 2, \quad X_2(3) = 1\end{aligned}$$

Assume we make the assessments $\underline{P}(X_1) = 2 = \underline{P}(X_2)$.

(a) Do these assessments avoid sure loss?

They do, because the linear prevision P associated with the mass function $(\frac{1}{2}, 0, \frac{1}{2})$ satisfies $P(X_1) = 2 = P(X_2)$, and as a consequence it dominates $\underline{P}$.

(b) Compute their natural extension on the gamble Y given by $Y(1) = 0, Y(2) = 1 = Y(3)$.

The credal set $\mathcal{P}(\underline{P})$ associated with $\underline{P}$ is given by the linear previsions P satisfying:

$$\begin{aligned}P(\{1\}) + 2P(\{2\}) + 3P(\{3\}) \geq 2\\ 3P(\{1\}) + 2P(\{2\}) + P(\{3\}) \geq 2\end{aligned}$$

These two equations are equivalent to $P(\{1\}) = P(\{3\})$. As a consequence, the extreme points of $\mathcal{P}(\underline{P})$ are

$$\left\{(\frac{1}{2}, 0, \frac{1}{2}), (0, 1, 0)\right\},$$

and from this we deduce that the natural extension of $\underline{P}$ is $\underline{E}(Y) = 0.5$.

7. Let $\underline{P}_A$ be the vacuous lower prevision relative to a set A, given by the assessment $\underline{P}_A(A) = 1$.

   (a) Does it avoid sure loss?

   Yes. It suffices to see that there exists some dominating linear prevision. Take $\omega \in A$ and consider the linear prevision P given by $P(X) = X(\omega)$ for any gamble X.

   (b) Prove that the natural extension $\underline{E}$ of $\underline{P}_A$ is equal to the vacuous lower prevision relative to A:

$$\underline{E}(X) = \underline{P}_A(X) = \inf_{\omega \in A} X(\omega),$$

   for any $X \in \mathcal{L}(()\Omega)$.

   $\underline{E}(X)$ is equal to the supremum achieved by the free variable μ subject to the constraint

$$X - \mu \geq \lambda(I_A - 1),$$

   where $\lambda \geq 0$. Note that $\mu = inf_{\omega \in X(\omega)}$ and $\lambda = \inf_{\omega \in A} X(\omega) - \inf_{\omega \in \Omega} X(\omega)$ yields a feasible solution this equation, whence $\underline{E}(X) \geq \inf_{\omega \in A} X(\omega)$.
   Consider next μ, λ any feasible solution of the equation above. Since $\inf_{\omega \in A}$ is monotone, we find in particular that $\inf_{\omega \in A} X(\omega) - \mu \geq \inf_{\omega \in A}(\lambda(I_A(\omega) - 1)) = 0$, whence $\mu \leq \inf_{\omega \in A} X(\omega)$. As a consequence, $\underline{E}(X) = \inf_{\omega \in A} X(\omega)$.

   (c) Is $\underline{P}_A$ coherent?

   Yes, since it coincides with its natural extension $\underline{E}$ on its domain $\{I_A\}$.

8. Consider a possibility space $\Omega := \{a, b, c\}$ and a set of gambles $\mathcal{K} := \{X_1, X_2\}$, whose values are given as column vectors in the table:

	X_1	X_2
a	1	0
b	1/2	1
c	0	1/2

   The so-called vacuous lower prevision $\underline{P}_A$ relative to a subset A of Ω is defined by $\underline{P}_A(X) := \min_{\omega \in \Omega} X(\omega)$ for any gamble X on Ω. We consider a number of lower previsions on $\mathcal{K}$ defined by

$$\underline{P}_1 := \underline{P}_\Omega, \qquad \underline{P}_2 := \underline{P}_{\{a,b\}}, \qquad \underline{P}_3 := \underline{P}_{\{a\}}, \qquad \underline{P}_4 := \tfrac{1}{3}(\underline{P}_{\{a\}} + \underline{P}_{\{b\}} + \underline{P}_{\{c\}}),$$

$$\underline{P}_5 := \tfrac{3}{4}\underline{P}_{\{c\}} + \tfrac{1}{2}\underline{P}_{\{b\}}, \qquad \underline{P}_6 := \tfrac{1}{3}\underline{P}_{\{c\}} + \tfrac{2}{3}\underline{P}_{\{a,b\}}, \qquad \underline{P}_7 := \tfrac{3}{4}\underline{P}_{\{b\}} + \tfrac{1}{2}\underline{P}_{\{a\}}.$$

   Write the values attained by these lower previsions in a table (so a value for X_1 and X_2 in each column).

	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$
X_1	0	1/2	1	1/2	1/4	1/3	7/8
X_2	0	0	0	1/2	7/8	1/6	3/4

9. Let $\underline{P}$ be the lower prevision on a linear space of gambles $\mathcal{L}(\Omega)$ given by

$$\underline{P}(X) := \frac{1}{2}(\min X + \max X)$$

   for all X on Ω. Is it coherent? (Hint: test superadditivity for functions that sum up to a constant.)

   No. Since $\underline{P}$ is defined on a linear space, a necessary condition for coherence is that it is supper-additive, meaning that $\underline{P}(X_1 + X_2) \geq \underline{P}(X_1) + \underline{P}(X_2)$ for any pair of gambles X_1, X_2. To see that this does not hold, consider $X_1 := -I_A$ and $X_2 := -I_{\Omega \setminus A}$. Then $\underline{P}(X_1) = \underline{P}(X_2) = -1/2$, while $\underline{P}(X_1 + X_2) = -1$.

10. Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\Omega)$, where $\Omega = \{0, 1\}$. Prove that $\underline{P}$ is a *linear-vacuous* mixture, i.e., that there is some $\alpha \in [0, 1]$ and a linear prevision P on Ω such that $\underline{P} = \alpha P + (1 - \alpha)\underline{P}_\Omega$.

   It follows from the coherence of $\underline{P}$ that for every gamble X on Ω,

$$\underline{P}(X) = \underline{P}(X - \min X) + \min X.$$

   Assume that $\underline{P}$ is non-vacuous (otherwise the result is trivial). Then it follows from the above equation that it must be either $\underline{P}(0) > 0$ or $\underline{P}(1) > 0$. Let $\alpha = \underline{P}(0) + \underline{P}(1)$, and let P be the linear prevision associated to the probability $P(0) = \frac{\underline{P}(0)}{\alpha}$ and $P(1) = \frac{\underline{P}(1)}{\alpha}$. Then for every gamble X it holds that

$$\alpha P(X) + (1 - \alpha)\underline{P}_\Omega(X) = \underline{P}(0)X(0) + \underline{P}(1)X(1) + (1 - \underline{P}(0) - \underline{P}(1)) \min X.$$

   Assume for instance that $\min X = X(0)$ (the other case is similar). Then

$$\begin{aligned}\alpha P(X) + (1 - \alpha)\underline{P}_\Omega(X) &= \underline{P}(0)X(0) + \underline{P}(1)X(1) + (1 - \underline{P}(0) - \underline{P}(1))X(0)\\ &= \underline{P}(1)(X(1) - X(0)) + X(0) = \underline{P}(X - X(0)) + X(0) = \underline{P}(X).\end{aligned}$$

11. Consider the lower prevision given by:

	$X(a)$	$X(b)$	$X(c)$	$\underline{P}(X)$
X_1	2	1	0	0.5
X_2	0	1	2	1
X_3	0	1	0	1

(a) Does it avoid sure loss? (Hint: Try to find a dominating linear prevision.)

Yes. It suffices to see that there is a linear prevision that dominates $\underline{P}$ on its domain. The prevision P given by $P(X) = \underline{P}_{\{b\}}(X) = X(b)$ satisfies this.

(b) Is it coherent? (Hint: Calculate the lower envelope of the set of linear previsions dominating $\underline{P}$.)

No. Since $\underline{P}(X_3) = 1 = \max\{X_3(a), X_3(b), X_3(c)\}$, the only linear prevision that dominates $\underline{P}$ for all X is precisely $P = \underline{P}_{\{b\}}$. But P does not coincide with $\underline{P}$ on all gambles: $P(X_1) = 1 > 0.5 = \underline{P}(X_1)$. Since $\underline{P}$ is not the lower envelope of the set $\mathcal{M}(\underline{P})$, we deduce that it is not coherent.

12. Consider the assessments

- $X_1(R) = 0,\ X_1(P) = 20,\ X_1(D) = -10,\ \underline{P}(X_1) = 0$
- $X_2(R) = 20,\ X_2(P) = 0,\ X_2(D) = -10,\ \underline{P}(X_2) = 0$
- $X_3(R) = 200, X_3(P) = 100, X_3(D) = 40, \underline{P}(X_3) = 120$

Answer these questions (in the order you deem fit)?

1. Do they avoid sure loss?

   Yes, for example the probability $p(R) = 1$ satisfy all constraints, hence the associated credal set is not empty.

2. If yes, What are the extreme points of the associated credal set?

   The extreme points are $\{(1, 0, 0), (0.2, 0.8, 0), (\frac{13}{21}, \frac{4}{21}, \frac{8}{21})\}$.

3. If yes to 1, Are they coherent?
4. If not to 3, which assessment(s) can be corrected?

   No, X_2 can be corrected: the natural extension is $\underline{X_2} = \frac{180}{21}$.

13. Mr. Play-it-safe is planning his upcoming holidays in the Canary Islands, and he is taking into account three possible disruptions: an unexpected illness (A), severe weather problems (B) and the unannounced visit of his mother in law (C).



He has assessed his lower and upper probabilities for these events:

	A	B	C	D
$\underline{P}$	0.05	0.05	0.2	0.5
$\overline{P}$	0.2	0.1	0.5	0.8

where D denotes the event 'Nothing bad happens'. He also assumes that no two disruptions can happen simultaneously.

(a) Determine the set of probabilities compatible with the assessments above.

The set of probabilities compatible with these assessments is given by

$$\mathcal{M} = \{P : P(A) \in [0.05, 0.2], P(B) \in [0.05, 0.1], P(C) \in [0.2, 0.8], P(D) \in [0.5, 0.8], P(A) + P(B) + P(C) + P(D) = 1\}.$$

The assumption that no two disruptions can happen simultaneously mean that events A, B, C, D are disjoint. The extreme points of this set are:

$$\{(0.05, 0.05, 0.2, 0.7), (0.05, 0.05, 0.4, 0.5), (0.2, 0.1, 0.2, 0.5)\}.$$

(b) Are these lower probabilities coherent? If not, compute their natural extension.

They are not coherent, because $\underline{P}, \overline{P}$ are not the lower and upper envelopes of the set of probabilities they determine: for instance, the upper envelope $\overline{E}$ of $\mathcal{M}$ satisfies $\overline{E}(D) = 0.7 < 0.8 = \overline{P}(D)$.
Their natural extension would be the coherent lower prevision $\underline{E}$ that we can obtain by taking the lower envelope of $\mathcal{M}$.

(c) What is the lower probability that something unexpected happens?

This would be $\underline{E}(A \cup B \cup C) = 1 - \overline{E}(D) = 0.3$.

14. Three horses (a,b and c) take part in a race. Our a priori lower probability for each horse being the winner is

$$\underline{P}(\{a\}) = 0.1, \quad \underline{P}(\{b\}) = 0.25, \quad \underline{P}(\{c\}) = 0.3,$$
$$\underline{P}(\{a,b\}) = 0.4, \quad \underline{P}(\{a,c\}) = 0.6, \quad \underline{P}(\{b,c\}) = 0.7.$$

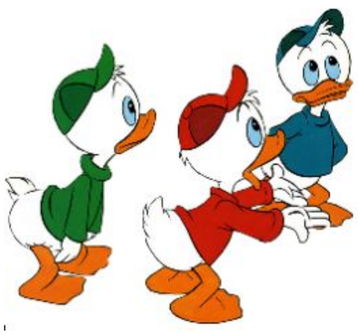
There are rumors that c is not going to take part in the race due to some injury. What are the updated lower probabilities for a, b?

Taking into account that we are dealing with finite spaces and that the conditioning event has positive lower probability, applying the Generalised Bayes Rule is equivalent to taking the lower envelope of the linear conditional previsions that we obtain applying Bayes's rule on the elements of $\mathcal{M}(\underline{P})$. Thus, we obtain:

$$\underline{P}(\{a\}|\{a,b\}) = \inf\left\{\frac{P(\{a\})}{P(\{a,b\})} : P \in \mathcal{M}(\underline{P})\right\} = 0.1/0.5 = 0.2,$$

$$\underline{P}(\{b\}|\{a,b\}) = \inf\left\{\frac{P(\{b\})}{P(\{a,b\})} : P \in \mathcal{M}(\underline{P})\right\} = 0.25/0.55 = 0.45.$$

15. Scrooge Mr Duck has been murdered in mysterious circumstances, and the police has narrowed the list of suspects to his great nephews Huey, Dewey and Louie.



They have gathered their evidence by means of the following assessments:

$$P(\text{Huey or Dewey}) \in [0.5, 0.8] \qquad P(\text{Huey or Louie}) \in [0.3, 0.6] \qquad P(\text{Louie or Dewey}) \in [0.6, 1].$$

However, after doing this it turns out that the night of the murder Louie was out all night with Daisy Duck.

(a) Determine the set of conditional probabilities for Huey and Dewey being the murderers according to this evidence.

Note that the assessments above are equivalent to stating that

$$P(\text{Louie}) \in [0.2, 0.5] \qquad P(\text{Dewey}) \in [0.4, 0.7] \qquad P(\text{Huey}) \in [0, 0.4].$$

If we denote by $\mathcal{M}$ the set of probabilities compatible with these assessments, we must compute

$$\{P(\cdot|\text{Huey or Dewey}) : P \in \mathcal{M}\}.$$

Taking into account that the extreme points of $\mathcal{M}$ are determined by the mass functions

$$\{(0, 0.5, 0.5), (0.4, 0.4, 0.2), (0.1, 0.4, 0.5), (0, 0.7, 0.3), (0.1, 0.7, 0.2)\}$$

where we are using the notation $(p_1, p_2, p_3) = (P(\text{Huey}), P(\text{Dewey}), P(\text{Louie}))$, the set of conditional probabilities has extreme points

$$\{(0, 1), (0.5, 0.5)\},$$

where we denote $(q_1, q_2) = (P(\text{Huey}|\text{Huey or Dewey}), P(\text{Dewey}|\text{Huey or Dewey}))$.

(b) Determine the conditional lower probability of Huey being the murderer.

From the previous statement, we deduce that $\underline{P}(\text{Huey}|\text{Huey or Dewey}) = 0$.

(c) Determine the conditional lower prevision of the gamble X given by $X(Huey) = 2, X(Dewey) = -3$.

It would be $\underline{P}(X|\text{Huey or Dewey}) = \min\{-3, -0.5\} = -3$.

16. **The three prisoners problem.** Three women, a, b and c, are in jail. Prisoner a knows that only two of the three prisoners will be executed, but she doesn't know who will be spared. She only knows that all three prisoners have equal probability $\frac{1}{3}$ of being spared. To the warden who knows which prisoner will be spared, a says, "Since two out of the three will be executed, it is certain that either b or c will be. You will give me no information about my own chances if you give me the name of one woman, b or c, who is going to be executed." Accepting this argument after some thinking, the warden says, "Prisoner b will be executed."

Does the warden's statement truly provide no information about the chance of a to be executed? We try to solve this problem using the theory of lower previsions.

(a) Determine the set of unconditional probabilities of each prisoner being spared (before talking to the warden).

According to the statement of the problem, we have $P(\{a\}) = P(\{b\}) = P(\{c\}) = \frac{1}{3}$.

(b) Determine the sets of conditional probabilities of the prisoner named by the warden, depending on the prisoner that will be spared.

Since the warden can only name b or c, we have that

$$\begin{aligned} P(\text{ names } c|\text{b is spared}) &= 1 \\ P(\text{ names } c|\text{c is spared}) &= 0 \\ P(\text{ names } c|\text{a is spared}) &\in [0,1], \end{aligned}$$

since when one of b or c is going to be spared the warden has only one choice (naming the other prisoner), and when a is going to be spared he could name any of them. We model our ignorance by allowing for any probability model in this case.

(c) Determine the set of joint probabilities by combining the marginal and conditional probabilities (the lower prevision they determine corresponds to what we call the *marginal extension*).

Let P be a probability model compatible with the information above, and let p denote the value of $P(\text{ names } c|\text{a is spared})$. Then it follows that P is determined by the following mass function:

$$\begin{aligned} P(\text{b is spared, the warden names c}) &= \frac{1}{3} \\ P(\text{c is spared, the warden names b}) &= \frac{1}{3} \\ P(\text{a is spared, the warden names c}) &= \frac{p}{3} \\ P(\text{a is spared, the warden names b}) &= \frac{1-p}{3}. \end{aligned}$$

(d) From this set of joint probabilities, compute the lower and upper probabilities of a being spared, given that the warden named prisoner b.

Let $\mathcal{M}$ be the set of probabilities determined in the previous statement. Then,

$$\{P(\text{a is spared}|\text{the warden names b}) : P \in \mathcal{M}\} = \left\{\frac{P(\text{a is spared, the warden names b})}{P(\text{the warden names b})} : P \in \mathcal{M}\right\}$$

$$= \left\{\frac{(1-p)/3}{(2-p)/3} : p \in [0,1]\right\} = \left[0, \frac{1}{2}\right].$$