Exercises for the 6th SIPTA Summer School

July 21-25, 2014 – Montpellier, France

1 Presentation examples

1. Are the following bets rational?

   - $$P(\{C\}) = 0.5, \quad P(\{C,P\}) = 0.3$$
   - $$P(\{C,P\}) = 0.5, \quad P(\{P,T\}) = 0.6, \quad P(\{C,T\}) = 0.7$$
   - $$P(\{C,P\}) = 0.5, \quad P(\{P,T\}) = 0.5$$

2. Are the following bets rational?

   (a)
   - $X_1(C) = 1$, $X_1(P) = -1$, $X_1(T) = 0$, $P(X_1) = 0$
   - $X_2(C) = 0$, $X_2(P) = 1$, $X_2(T) = -1$, $P(X_2) = 0$
   - $X_3(C) = 1$, $X_3(P) = -1$, $X_3(T) = 1$, $P(X_3) = 0$

   (b)
   - $X_1(C) = 2$, $X_1(P) = -1$, $X_1(T) = 0$, $P(X_1) = 1$
   - $X_2(C) = 1$, $X_2(P) = 1$, $X_2(T) = -1$, $P(X_2) = 0$

3. Are the following bets rational?

   (a)
   - $X_1(R) = 20$, $X_1(P) = -40$, $X_1(D) = 60$, $\underline{P}(X_1) = 20$
   - $X_2(R) = -20$, $X_2(P) = 40$, $X_2(D) = -60$, $\underline{P}(X_2) = -10$

   (b)
   - $X_1(R) = 10$, $X_1(P) = -20$, $X_1(D) = 0$, $\underline{P}(X_1) = 0$
   - $X_2(R) = 0$, $X_2(P) = 10$, $X_2(D) = 0$, $\underline{P}(X_2) = 5$

   (c)
   - $X_1(R) = 0$, $X_1(P) = 20$, $X_1(D) = -10$, $\underline{P}(X_1) = 0$
   - $X_2(R) = 20$, $X_2(P) = 0$, $X_2(D) = -10$, $\underline{P}(X_2) = 0$

2 Exercises

1. Form groups of 2/3 students, each person assessing separately price for **one (or two) different** gambles among the three following ones that give 1 if events/assertions are true: Montpellier agglomeration (city + neighbouring villages) counts between

   - 300K and 400K inhabitants
   - 400K and 500K inhabitants
   - 500K and 600K inhabitants

   Compare your assessments. Discuss how you did to reach such prices? Are they rational? Can you agree on rational prices?

2. Represent the assessments
   - $X_1(C) = 1,\ X_1(P) = -1,\ X_1(T) = 0,\ P(X_1) = 0$
   - $X_2(C) = 0,\ X_2(P) = 1,\ X_2(T) = -1,\ P(X_2) = 0$
   - $X_3(C) = 1,\ X_3(P) = -1,\ X_3(T) = 1,\ P(X_3) = 0$

   and

   - $X_1(C) = 2,\ X_1(P) = -1,\ X_1(T) = 0,\ P(X_1) = 1$
   - $X_2(C) = 1,\ X_2(P) = 1,\ X_2(T) = -1,\ P(X_2) = 0$

   In a geometric way, and check in this way that they are rational or not

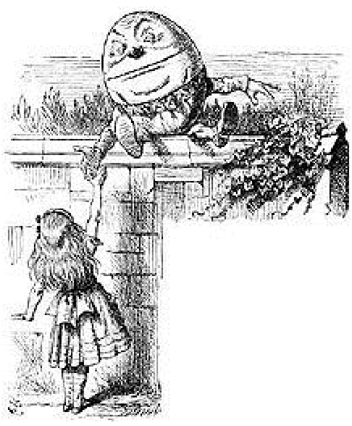
3. John is planning to bet on the winner of the Formula 1 championship.

   (a) Determine the set of probabilities compatible with his beliefs, if he thinks that:
      – Only one of Rosberg, Hamilton, Alonso or Vettel can win.
      – The probability of Hamilton winning is at least twice as much of that of Alonso winning, and this is at least 1.5 times the probability of Vettel winning.
      – Rosberg has exactly the same probability of winning than Hamilton.

   (b) They offer him a bet with reward 10 if Alonso wins, 5 if Vettel wins, and -3 if either Rosberg or Hamilton win. According to his beliefs, which are the minimum and maximum expected gains? Should he accept this bet or not?

4. Before jumping off the wall, Humpty Dumpty tells Alice the following:

   > “I have a farm with pigs, cows and hens. There are at least as many pigs as cows and hens together, and at least as many hens as cows. How many pigs, cows and hens do I have?”



   (a) Determine the set of probabilities compatible with these assessments.

   (b) Can we express them using lower and upper probabilities of events?

Before going away, Humpty Dumpty tells Alice the following:

> I see you have some money with you. If you give me a coin, we will wait till an animal come out of the barn, and:
>
> - I will give you 1 if a pig comes out
> - I will give you 4 if a hen comes out
> - you will give me 1 if a cow comes out

Will Alice accept the gamble? Would she have accepted if she retained only the lower/upper probabilities?

5. Consider an urn with 10 balls, of which 3 are red, and the other 7 are either blue or yellow.
   (a) Determine the set $\mathcal{M}$ of linear previsions that represent the possible compositions of the urn.
   (b) Which are the extreme points of this set?
   (c) What are the lower and upper probabilities that the ball selected is blue?
   (d) Let X be a gamble given by $X(\text{blue}) = 50, X(\text{red}) = 100, X(\text{yellow}) = 300$. What is the lower prevision of X?
   (e) Do the same for an arbitrary gamble Y.

6. Consider $\Omega = \{1, 2, 3\}$, and the gambles and lower previsions given by

$$X_1(1) = 1, \quad X_1(2) = 2, \quad X_1(3) = 3$$
$$X_2(1) = 3, \quad X_2(2) = 2, \quad X_3(3) = 1$$

   Assume we make the assessments $\underline{P}(X_1) = 2 = \underline{P}(X_2)$.
   (a) Do these assessments avoid sure loss?
   (b) Compute their natural extension on the gamble Y given by $Y(1) = 0, Y(2) = 1 = Y(3)$.

7. Let $\underline{P}_A$ be the vacuous lower prevision relative to a set A, given by the assessment $\underline{P}_A(A) = 1$.
   (a) Does it avoid sure loss?
   (b) Prove that the natural extension $\underline{E}$ of $\underline{P}_A$ is equal to the vacuous lower prevision relative to A:

$$\underline{E}(X) = \underline{P}_A(X) = \inf_{\omega \in A} X(\omega),$$

   for any $X \in \mathcal{L}(\Omega)$.
   (c) Is $\underline{P}_A$ coherent?

8. Consider a possibility space $\Omega := \{a, b, c\}$ and a set of gambles $\mathcal{K} := \{X_1, X_2\}$, whose values are given as column vectors in the table:

	X_1	X_2
a	1	0
b	1/2	1
c	0	1/2

   The so-called vacuous lower prevision $\underline{P}_A$ relative to a subset A of Ω is defined by $\underline{P}_A(X) := \min_{\omega \in \Omega} X(\omega)$ for any gamble X on Ω. We consider a number of lower previsions on $\mathcal{K}$ defined by

$$\underline{P}_1 := \underline{P}_{\mathcal{X}}, \qquad \underline{P}_2 := \underline{P}_{\{a,b\}}, \qquad \underline{P}_3 := \underline{P}_{\{a\}}, \qquad \underline{P}_4 := \tfrac{1}{3}(\underline{P}_{\{a\}} + \underline{P}_{\{b\}} + \underline{P}_{\{c\}}),$$
$$\underline{P}_5 := \tfrac{3}{4}\underline{P}_{\{c\}} + \tfrac{1}{2}\underline{P}_{\{b\}}, \qquad \underline{P}_6 := \tfrac{1}{3}\underline{P}_{\{c\}} + \tfrac{2}{3}\underline{P}_{\{a,b\}}, \qquad \underline{P}_7 := \tfrac{3}{4}\underline{P}_{\{b\}} + \tfrac{1}{2}\underline{P}_{\{a\}}.$$

   Write the values attained by these lower previsions in a table (so a value for X_1 and X_2 in each column).

9. Let $\underline{P}$ be the lower prevision on a linear space of gambles $\mathcal{L}(\Omega)$ given by

$$\underline{P}(X) := \frac{1}{2}(\min X + \max X)$$

for all X on Ω. Is it coherent? (Hint: test superadditivity for functions that sum up to a constant.)

10. Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\Omega)$, where $\Omega = \{0, 1\}$. Prove that $\underline{P}$ is a *linear-vacuous* mixture, i.e., that there is some $\alpha \in [0, 1]$ and a linear prevision P on Ω such that $\underline{P} = \alpha P + (1 - \alpha)\underline{P}_\Omega$.

11. Consider the lower prevision given by:

	$X(a)$	$X(b)$	$X(c)$	$\underline{P}(X)$
X_1	2	1	0	0.5
X_2	0	1	2	1
X_3	0	1	0	1

(a) Does it avoid sure loss? (Hint: Try to find a dominating linear prevision.)

(b) Is it coherent? (Hint: Calculate the lower envelope of the set of linear previsions dominating $\underline{P}$.)

12. Consider the assessments

- $X_1(R) = 0$, $X_1(P) = 20$, $X_1(D) = -10$, $\underline{P}(X_1) = 0$
- $X_2(R) = 20$, $X_2(P) = 0$, $X_2(D) = -10$, $\underline{P}(X_2) = 0$
- $X_3(R) = 200, X_3(P) = 100, X_3(D) = 40, \underline{P}(X_3) = 120$

Answer these questions (in the order you deem fit)?

1. Do they avoid sure loss?
2. If yes, What are the extreme points of the associated
3. If yes to 1, Are they coherent?
4. If not to 3, which assessment(s) can be corrected?

13. Mr. Play-it-safe is planning his upcoming holidays in the Canary Islands, and he is taking into account three possible disruptions: an unexpected illness (A), severe weather problems (B) and the unannounced visit of his mother in law (C).



He has assessed his lower and upper probabilities for these events:

	A	B	C	D
$\underline{P}$	0.05	0.05	0.2	0.5
$\overline{P}$	0.2	0.1	0.5	0.8

where D denotes the event ‘Nothing bad happens’. He also assumes that no two disruptions can happen simultaneously.

(a) Determine the set of probabilities compatible with the assessments above.

(b) Are these lower probabilities coherent? If not, compute their natural extension.

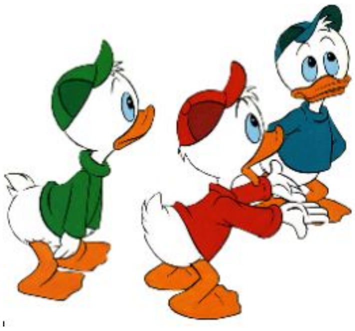
(c) What is the lower probability that something unexpected happens?

14. Three horses (a,b and c) take part in a race. Our a priori lower probability for each horse being the winner is

$$\underline{P}(\{a\}) = 0.1, \quad \underline{P}(\{b\}) = 0.25, \quad \underline{P}(\{c\}) = 0.3,$$
$$\underline{P}(\{a,b\}) = 0.4, \quad \underline{P}(\{a,c\}) = 0.6, \quad \underline{P}(\{b,c\}) = 0.7.$$

There are rumors that c is not going to take part in the race due to some injury. What are the updated lower probabilities for a, b?

15. Scrooge Mr Duck has been murdered in mysterious circumstances, and the police has narrowed the list of suspects to his great nephews Huey, Dewey and Louie.



They have gathered their evidence by means of the following assessments:

$$P(\text{Huey or Dewey}) \in [0.5, 0.8] \quad P(\text{Huey or Louie}) \in [0.3, 0.6] \quad P(\text{Louie or Dewey}) \in [0.6, 1].$$

However, after doing this it turns out that the night of the murder Louie was out all night with Daisy Duck.

(a) Determine the set of conditional probabilities for Huey and Dewey being the murderers according to this evidence.
(b) Determine the conditional lower probability of Huey being the murderer.
(c) Determine the conditional lower prevision of the gamble X given by $X(Huey) = 2, X(Dewey) = -3$.

16. **The three prisoners problem.** Three women, a, b and c, are in jail. Prisoner a knows that only two of the three prisoners will be executed, but she doesn't know who will be spared. She only knows that all three prisoners have equal probability $\frac{1}{3}$ of being spared. To the warden who knows which prisoner will be spared, a says, "Since two out of the three will be executed, it is certain that either b or c will be. You will give me no information about my own chances if you give me the name of one man, b or c, who is going to be executed." Accepting this argument after some thinking, the warden says, "Prisoner b will be executed."

Does the warden's statement truly provide no information about the chance of a to be executed? We try to solve this problem using the theory of lower previsions.

(a) Determine the set of unconditional probabilities of each prisoner being executed (before talking to the warden).
(b) Determine the sets of conditional probabilities of the prisoner named by the warden, depending on the prisoner that will be spared.
(c) Determine the set of joint probabilities by combining the marginal and conditional probabilities (the lower prevision they determine corresponds to what we call the *marginal extension*).
(d) From this set of joint probabilities, compute the lower and upper probabilities of a being spared, given that the warden named prisoner b.