1. Two notions of symmetry

We want to model aspects of symmetry using imprecise probability models.

We are uncertain about something: the value that a variable X assumes in a *finite* set $\mathcal{X}$.

The models we will use are coherent lower previsions $\underline{P}$, or equivalently, convex closed sets $\mathcal{M}$ of mass functions p—also called *credal sets*.

Recall from previous lectures: a (probability) *mass function* p on $\mathcal{X}$ is a real-valued map on $\mathcal{X}$ such that

$$(\forall x \in \mathcal{X})p(x) \geq 0 \text{ and } \sum_{x\in\mathcal{X}} p(x) = 1.$$

We denote the set (simplex) of all mass functions on *values* by $\Sigma_{\mathcal{X}}$.

With a mass function p there corresponds an *expectation operator* E_p defined on the set $\mathcal{L}(\mathcal{X})$ of all gambles on $\mathcal{X}$:

$$E_p(f) \coloneqq \sum_{x\in\mathcal{X}} p(x)f(x) \text{ for all } f\colon \mathcal{X} \to \mathbb{R}.$$

A *coherent lower prevision* on $\mathcal{L}(\mathcal{X})$ is a map $\mathcal{L}(\mathcal{X}) \to \mathbb{R}$ with the following properties:

P1. $\underline{P}(f) \geq \min f$ for all $f \in \mathcal{L}(\mathcal{X})$ [bounds]

P2. $\underline{P}(f + g) \geq \underline{P}(f) + \underline{P}(g)$ for all $f, g \in \mathcal{L}(\mathcal{X})$ [super-additivity]

P3. $\underline{P}(\lambda f) = \lambda\underline{P}(f)$ for all $f \in \mathcal{L}(\mathcal{X})$ and all real $\lambda \geq 0$ [non-negative homogeneity]

Its *conjugate upper prevision* is defined by

$$\overline{P}(f) \coloneqq -\underline{P}(-f) \text{ for all } f \in \mathcal{L}(\mathcal{X}).$$

With a convex closed set $\mathcal{M}$ of mass functions we can construct a coherent lower prevision on $\mathcal{L}(\mathcal{X})$ by

$$\underline{P}(f) \coloneqq \min\{E_p(f) \colon p \in \mathcal{M}\} \text{ for all } f\colon \mathcal{X} \to \mathbb{R},$$

and the conjugate upper prevision on $\mathcal{L}(\mathcal{X})$ by

$$\overline{P}(f) \coloneqq \max\{E_p(f) \colon p \in \mathcal{M}\} \text{ for all } f\colon \mathcal{X} \to \mathbb{R}.$$

Conversely, with a coherent lower prevision $\underline{P}$ there corresponds a convex closed set of mass functions, given by

$$\mathcal{M} \coloneqq \{p \in \Sigma_{\mathcal{X}} \colon (\forall f \in \mathcal{L}(\mathcal{X}))E_p(f) \geq \underline{P}(f)\}.$$

A precise model is a singleton $\mathcal{M} = \{p\}$ and the associated lower prevision is the (self-conjugate) expectation operator E_p.

Symmetry is typically modelled by considering a collection of transformations of the space of interest, and something is considered to be symmetrical when it is left unchanged by these transformations.

Here, we will focus on a *group* $\mathcal{P}$ of permutations π of $\mathcal{X}$, meaning that:

G1. $\pi_1 \circ \pi_2 \in \mathcal{P}$ for all $\pi_1, \pi_2 \in \mathcal{P}$ [[]internality]

G2. $\pi_1 \circ (\pi_2 \circ \pi_3) = (\pi_1 \circ \pi_2) \circ \pi_3$ for all $\pi_1, \pi_2, \pi_3 \in \mathcal{P}$ [associativity]

G3. $\pi \circ \mathrm{id} = \mathrm{id} \circ \pi$ for all $\pi \in \mathcal{P}$ [neutral element]

G4. For all $\pi \in \mathcal{P}$ there is some $\pi^{-1} \in \mathcal{P}$ such that $\pi \circ \pi^{-1} = \pi^{-1} \circ \pi = \mathrm{id}$ [inverse]

We will use the simpler notation $\varpi\pi \coloneqq \varpi \circ \pi$.

Our uncertainty models involve gambles, so we need a way to let permutations act on gambles. This is easy to do by a procedure called *lifting*: for any $\pi \in \mathcal{P}$, $\pi^t f \coloneqq f \circ \pi$, meaning that

$$(\pi^t f)(x) \coloneqq f(\pi x) \text{ for all } x \in \mathcal{X}.$$

Running example. We flip a coin twice, and the uncertain outcome X is an element of the finite set $\mathcal{X} = \{HH, HT, TH, TT\}$.

The symmetry we consider is that the order of the observations does not matter, which leads us to identify HT with TH. This symmetry is embodied in the group of permutations of $\mathcal{X}$ given by

$$\mathcal{P} = \{\mathrm{id}, \varpi\},$$

where ϖ is the permutation defined by

$$\begin{pmatrix} HH & HT & TH & TT \\ HH & TH & HT & TT \end{pmatrix}$$

Observe that this a group, with $\varpi^2 = \varpi \circ \varpi = \mathrm{id}$, so $\varpi^{-1} = \varpi$.

Consider the gamble f that yields a gain of 2 when the two coin flips produce the same result, and a loss of 1 otherwise:

$$f = 2I_{\{HH,TT\}} - I_{\{HT,TH\}}.$$

Observe that $\varpi^t f = f$.

On the other, for the gamble g that yields a gain of 1 when the first coin flip produces H, and a loss of 3 otherwise:

$$g = I_{\{HH,HT\}} - 3I_{\{TH,TT\}},$$

the permuted gamble

$$\varpi^t g = I_{\{HH,TH\}} - 3I_{\{HT,TT\}}$$

yields a gain of 1 when the second coin flip produces H, and a loss of 3 otherwise. □

<u>Important</u>: The set $\mathcal{L}(\mathcal{X})$ of all $\mathcal{X} - \mathbb{R}$ maps can also be written as $\mathbb{R}^{\mathcal{X}}$, which reminds us that any gamble f can be seen as a vector in the $|\mathcal{X}|$-dimensional *linear space* $\mathcal{L}(\mathcal{X})$, with as many components $f(x)$ as there are values x in $\mathcal{X}$.

Any lifted permutation π^t is a linear permutation (a linear isomorphism) of $\mathcal{L}(\mathcal{X})$ (check this):

$$\pi^t(\lambda f + \mu g) = \lambda \pi^t f + \mu \pi^t g \text{ for all } f, g \in \mathcal{L}(\mathcal{X}) \text{ and } \lambda, \mu \in \mathbb{R}.$$

This suggests that a geometrical way of looking at gambles and permutations may be fruitful.

For any linear transformation T of $\mathcal{L}(\mathcal{X})$, its *range* $\operatorname{rng} T$ is the linear subspace of gambles that it reaches:

$$\operatorname{rng} T \coloneqq T(\mathcal{L}(\mathcal{X})) = \{Tf \colon f \in \mathcal{L}(\mathcal{X})\}.$$

Its *kernel* $\operatorname{kern} T$ is the linear subspace of gambles that it maps to the zero gamble:

$$\operatorname{kern} T \coloneqq \{f \in \mathcal{L}(\mathcal{X}) \colon Tf = 0\}.$$

And for any subset $\mathcal{C}$ of $\mathcal{L}(\mathcal{X})$, its *linear span*

$$\operatorname{span}(\mathcal{C}) \coloneqq \left\{ \sum_{k=1}^{n} \lambda_k f_k \colon n \geq 0, f_k \in \mathcal{C}, \lambda_k \in \mathbb{R} \right\}$$

is the smallest linear subspace that includes $\mathcal{C}$.

Now there are two fundamentally different concepts of symmetry for imprecise probability models.

Symmetrical models. The first one captures that the uncertainty models $\mathcal{M}$ or $\underline{P}$ we are using, are symmetrical, or in other words, invariant under the permutations in $\mathcal{P}$. This means that

$$(\forall f \in \mathcal{L}(\mathcal{X}))\underline{P}(f) = \underline{P}(\pi^t f) \text{ or equivalently } \underline{P} = \underline{P} \circ \pi^t, \text{ for all } \pi \in \mathcal{P}.$$

This requirement for lower previsions is equivalent (check this) to the following requirement for credal sets:

$$(\forall p \in \mathcal{M})\pi^t p \in \mathcal{M} \text{ or equivalently } \pi^t(\mathcal{M}) = \mathcal{M}, \text{ for all } \pi \in \mathcal{P},$$

where, as before, we let $\pi^t p := p \circ \pi$. We then say that the models $\underline{P}$ and $\mathcal{M}$ are (weakly) *invariant* (with respect to $\mathcal{P}$).

This type of symmetry is the one that is usually invoked under ignorance. When we are ignorant, our models ought to be symmetrical under all kinds of permutations. Observe in this respect that the *vacuous model*

$$\underline{P}_v := \min \text{ or equivalently } \mathcal{M}_v := \Sigma_{\mathcal{X}}$$

is symmetrical under all permutations of $\mathcal{X}$.

A precise model $\mathcal{M} = \{p\}$, or equivalently, a self-conjugate lower prevision (expectation operator) E_p, is therefore invariant if and only if

$$\pi^t p = p \text{ or equivalently } E_p \circ \pi^t = E_p, \text{ for all } \pi \in \mathcal{P}.$$

Running example. Consider the mass functions

$$p_1 := \frac{1}{3} I_{\{HH,HT\}} + \frac{1}{6} I_{\{TH,TT\}} \text{ and } p_2 := \frac{1}{3} I_{\{TH,HH\}} + \frac{1}{6} I_{\{HT,TT\}}$$

For both precise models, the coin flips are independent. In the first model, the second coin is fair and the probability of heads for the first coin flip is $\frac{2}{3}$. The second model is similar to the first, but the role of the first and second coin flips is reversed. Observe that $\varpi^t p_1 = p_2$ and consequently also $\varpi^t p_2 = p_1$, so these precise models are *not* permutation invariant with respect to $\mathcal{P}$. On the other hand, the credal set

$$\mathcal{M}_1 := \{\alpha p_1 + (1-\alpha)p_2 \colon \alpha \in [0,1]\}$$

is permutation invariant, because

$$\varpi^t[\alpha p_1 + (1-\alpha)p_2] = \alpha\varpi^t p_1 + (1-\alpha)\varpi^t p_2 = \alpha p_2 + (1-\alpha)p_1 \in \mathcal{M}_1.$$

This model corresponds to the following probability assessments: the structural assessment that the coin flips are (strongly) independent, and the local assessments that the (precise) probability of HH is $\frac{1}{3}$ and the (precise) probability of TT is $\frac{1}{6}$. Instead of the independence assessment, we can also simply require that the probability of HT lies between $\frac{1}{6}$ and $\frac{1}{3}$.

Observe that this model is equivalent to assuming that two different coins are used for each coin flip, but that there is no information at all about the mechanism for selecting the order in which these two coins are used. It is this lack of information (not the symmetry of the coins—they aren't) that is reflected in the symmetry of the model. □

Models of symmetry. There is a (usually much) stronger requirement, called *strong invariance*, that is used to model that there is symmetry lurking behind the variable X, or that the subject believes that the 'mechanism generating the observations of the variable X is symmetrical'.

How can this be modelled? Consider any gamble f. If the subject believes there is this symmetry, he will be indifferent between f and its permutations $\pi^t f$, for all $\pi \in \mathcal{P}$. This implies that he is willing to exchange f for $\pi^t f$ when he is paid any positive amount of utility $\epsilon > 0$, whence

$$\underline{P}(\pi^t f - f + \epsilon) \geq 0 \text{ for all } \epsilon > 0,$$

so the requirement for *strong invariance* (with respect to $\mathcal{P}$) becomes

$$\underline{P}(\pi^t f - f) = \overline{P}(\pi^t f - f) = 0 \text{ for all } f \in \mathcal{L}(\mathcal{X}) \text{ and all } \pi \in \mathcal{P}.$$

This requirement for lower previsions $\underline{P}$ is equivalent (check this) to the following requirement for credal sets $\mathcal{M}$:

$$(\forall p \in \mathcal{M})\pi^t p = p, \text{ for all } \pi \in \mathcal{P}.$$

So a lower prevision is strongly invariant if and only if it is a lower envelope of (weakly and therefore strongly) invariant precise expectations.

Strong invariance is indeed a stronger requirement than invariance: any strongly invariant model is invariant (check this), but not every invariant model is strongly invariant. Check, for instance, that the vacuous model is not strongly invariant with respect to any non-trivial group of permutations.

Running example. Consider the mass functions

$$p_3 := \frac{1}{3} I_{\{HH,TT\}} + \frac{1}{6} I_{\{HT,TH\}} \text{ and } p_4 := \frac{1}{6} I_{\{HH,TT\}} + \frac{1}{3} I_{\{HT,TH\}}$$

For neither of these precise models, the coin flips are independent. Observe that $\varpi^t p_3 = p_3$ and $\varpi^t p_4 = p_4$, so both these precise models are permutation invariant with respect to $\mathcal{P}$. This implies that the credal set

$$\mathcal{M}_2 := \{\alpha p_3 + (1-\alpha) p_4 \colon \alpha \in [0,1]\}$$

is *strongly* permutation invariant, because

$$\varpi^t[\alpha p_3 + (1-\alpha)p_4] = \alpha \varpi^t p_3 + (1-\alpha)\varpi^t p_4 = \alpha p_3 + (1-\alpha) p_4.$$

The lower prevision that corresponds to this credal set if given by:

$$\underline{P}_2(f) := \min\Big\{\frac{1}{3}[f(HH)+f(TT)]+\frac{1}{6}[f(HT)+f(TH)], \frac{1}{6}[f(HH)+f(TT)]+\frac{1}{3}[f(HT)+f(TH)]\Big\}$$

The independent and permutation invariant precise models are given by

$$p_r := r^2 I_{\{HH\}} + r(1-r) I_{\{HT,TH\}} + (1-r)^2 I_{\{TT\}}, \text{ for } r \in [0,1].$$

This implies that, for instance, the credal set

$$\mathcal{M}_{r_1,r_2} := \{\alpha p_{r_1} + (1-\alpha) p_{r_2} \colon \alpha \in [0,1]\}$$

is strongly independent and permutation invariant, for any choice of $r_1, r_2 \in [0,1]$. □

2. A general representation theorem

We will now show that, because of their symmetry properties, the behaviour of strongly invariant lower previsions can be represented more efficiently and compactly on lower-dimensional spaces than $\mathcal{L}(\mathcal{X})$. Our argumentation relies quite heavily on the geometrical interpretation of gambles as vectors in a linear space, and of permutations as linear transformations of that space.

Technical preliminaries about permutation groups. A gamble f is invariant if it is left unchanged by the permutations, so

$$\pi^t f = f \text{ for all } \pi \in \mathcal{P}.$$

An event $A \subseteq \mathcal{X}$ is invariant if its indicator I_A is, meaning that (check this)

$$(\forall x \in A)\pi x \in A \text{ or equivalently } \pi(A) = A, \text{ for all } \pi \in \mathcal{P}.$$

The smallest invariant sets are the so-called *invariant atoms*

$$[x] := \{\pi x \colon \pi \in \mathcal{P}\},$$

which constitute a partition of $\mathcal{X}$. We denote the set of all invariant atoms by $\mathcal{A}_{\mathcal{P}}$.

A gamble is permutation invariant if and only if it is constant on the invariant atoms (check). So a permutation invariant gamble is completely determined by the values it assumes on the invariant atoms, and this means that we can consider a one-to-one map $C_{\mathcal{P}}$—a linear isomorphiam—linking gambles g on the atoms, so $g \in \mathcal{L}(\mathcal{A}_{\mathcal{P}})$, to the corresponding permutation invariant gambles $C_{\mathcal{P}}g$ in $\mathcal{L}_{\mathcal{P}}(\mathcal{X})$, defined by:

$$C_{\mathcal{P}} \colon \mathcal{L}(\mathcal{A}_{\mathcal{P}}) \to \mathcal{L}_{\mathcal{P}}(\mathcal{X}), \text{ where } (C_{\mathcal{P}}g)(x) \coloneqq g([x]) \text{ for all } x \in \mathcal{X}.$$

Running example. The invariant atoms are

$$[HH] = \{HH\} \text{ and } [HT] = [TH] = \{HT, TH\} \text{ and } [TT] = \{TT\}.$$

The permutation invariant gambles are the ones that are constant on $[HT] = [TH] = \{HT, TH\}$ and therefore give the same value to HT and TH. □

Two interesting subspaces. We will need to look at two special linear subspaces. First, consider the following linear subspace:

$$\mathcal{I}_{\mathcal{P}} \coloneqq \operatorname{span}(\{\pi^t f - f \colon f \in \mathcal{L}(\mathcal{X}) \text{ and } \pi \in \mathcal{P}\}).$$

It is easy to check (do this) that $\underline{P}$ is strongly invariant if and only if

$$\underline{P}(g) = \overline{P}(g) = 0 \text{ for all } g \in \mathcal{I}_{\mathcal{P}},$$

so we can see $\mathcal{I}_{\mathcal{P}}$ as the linear subspace of *indifferent* gambles: the gambles that the subject judges to be equivalent to the zero gamble.

On the other hand, the linear subspace of permutation invariant gambles

$$\mathcal{L}_{\mathcal{P}}(\mathcal{X}) \coloneqq \{f \in \mathcal{L}(\mathcal{X}) \colon (\forall \pi \in \mathcal{P}) \pi^t f = f\}$$

is the set of all gambles that are constant on the invariant atoms. They are completely determined by the values that they assume on these invariant atoms, and the dimension of this space $\mathcal{L}_{\mathcal{P}}(\mathcal{X})$ is therefore the same as the dimension of the linear space $\mathcal{L}(\mathcal{A}_{\mathcal{P}})$ that linearly isomorphic to it, and therefore equal to the number of invariant atoms. This is generally smaller than the dimension $|\mathcal{X}|$ of the original space $\mathcal{L}(\mathcal{X})$: typically, the more permutations there are in $\mathcal{P}$, the fewer invariant atoms there are.

Running example. The subspace of indifferent gambles is given by:

$$\mathcal{I}_{\mathcal{P}} = \big\{\lambda(I_{\{HT\}} - I_{\{TH\}}) \colon \lambda \in \mathbb{R}\big\},$$

and is one-dimensional. The subspace of permutation invariant gambles is given by

$$\mathcal{L}_{\mathcal{P}}(\mathcal{X}) = \big\{\lambda_1 I_{\{HH\}} + \lambda_2 I_{\{TH,HT\}} + \lambda_3 I_{\{TT\}} \colon \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\big\},$$

and has dimension 3. Observe that $\mathcal{I}_{\mathcal{P}} \cap \mathcal{L}_{\mathcal{P}}(\mathcal{X}) = \{0\}$, and that

$$\begin{aligned}\mathcal{I}_{\mathcal{P}} + \mathcal{L}_{\mathcal{P}}(\mathcal{X}) &= \big\{\lambda_1 I_{\{HH\}} + \lambda_2 I_{\{TH,HT\}} + \lambda_3 I_{\{TT\}} + \lambda(I_{\{HT\}} - I_{\{TH\}}) \colon \lambda, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\big\} \\ &= \big\{\lambda_1 I_{\{HH\}} + (\lambda_2 - \lambda) I_{\{TH\}} + (\lambda_2 + \lambda) I_{\{HT\}} + \lambda_3 I_{\{TT\}} \colon \lambda, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\big\} \\ &= \mathcal{L}(\mathcal{X}),\end{aligned}$$

so any gamble can be decomposed uniquely into an indifferent and a permutation invariant part. This is a result that holds in general, and underlies the representation result we prove below. It is related to—a special case of—the famous dimension theorem of linear algebra. □

An interesting projection operator. Consider the following operator inv, which maps any gamble to the uniform average of all its permutations:

$$\operatorname{inv}_{\mathcal{P}} f \coloneqq \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}} \pi^t f.$$

Check that this is a linear transformation of $\mathcal{L}(\mathcal{X})$ that satisfies the following properties:

I1. $\operatorname{inv}_{\mathcal{P}} \circ \pi^t = \operatorname{inv}_{\mathcal{P}} = \pi^t \circ \operatorname{inv}_{\mathcal{P}}$ for all $\pi \in \mathcal{P}$ [permutation invariance]
I2. $\operatorname{inv}_{\mathcal{P}} \circ \operatorname{inv}_{\mathcal{P}} = \operatorname{inv}_{\mathcal{P}}$ [projection]
I3. $\operatorname{kern}(\operatorname{inv}_{\mathcal{P}}) = \mathcal{I}_{\mathcal{P}}$ [kernel]
I4. $\operatorname{rng}(\operatorname{inv}_{\mathcal{P}}) = \mathcal{L}_{\mathcal{P}}(\mathcal{X})$ [range]

So we see that this operator does not distinguish between gambles and their permuted versions. It is a projection operator that maps any gamble f to a permutation invariant gamble $\operatorname{inv}_{\mathcal{P}} f$ that the subject is indifferent between, in the sense that $f - \operatorname{inv}_{\mathcal{P}} f \in \mathcal{I}_{\mathcal{P}}$.

Since $\operatorname{inv}_{\mathcal{P}} f$ is a permutation invariant gambles, it is constant on the invariant atoms. The constant value it assumes there can also be written as:

$$(\operatorname{inv}_{\mathcal{P}} f)(x) = \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}} f(\pi x) = \frac{1}{|[x]|} \sum_{y \in [x]} f(y) \eqqcolon U(f|[x]) \text{ for all } x \in \mathcal{X},$$

which is the expectation associated with the uniform distribution over the atom $[x]$. We can see U as a linear map taking gambles f on $\mathcal{X}$ to the corresponding gambles $U(f|\cdot)$ on $\mathcal{A}_{\mathcal{P}}$:

$$U\colon \mathcal{L}(\mathcal{X}) \to \mathcal{L}(\mathcal{A}_{\mathcal{P}}), \text{ where } U(f)([x]) \coloneqq U(f|[x]) = \frac{1}{|[x]|} \sum_{y \in [x]} f(y),$$

which allows us to write

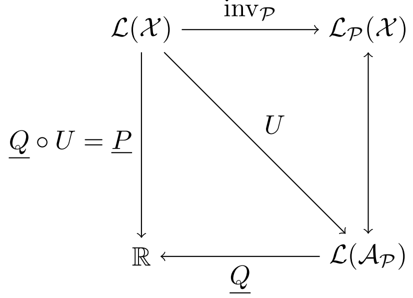
$$\operatorname{inv}_{\mathcal{P}} = C_{\mathcal{P}} \circ U.$$

This allows us to prove (and the proof is very easy) the following important result:

Theorem (Strong Permutation Invariance Representation Theorem)**.** *A coherent lower prevision $\underline{P}$ is strongly invariant with respect to $\mathcal{P}$ if and only if any (and hence all) of the following equivalent statements holds:*

(i) $\underline{P} = \underline{P} \circ \operatorname{inv}_{\mathcal{P}}$*;*
(ii) *There is a coherent lower prevision $\underline{Q}$ on $\mathcal{L}(\mathcal{A}_{\mathcal{P}})$ such that $\underline{P} = \underline{Q} \circ U$.*

In that case this so-called representing *lower prevision $\underline{Q}$ is unique and given by $\underline{Q} \coloneqq \underline{P} \circ C_{\mathcal{P}}$.*



Running example. With $\mathcal{P} = \{\mathrm{id}, \varpi\}$, we have $\mathrm{inv}_{\mathcal{P}} = \frac{1}{2}(\mathrm{id} + \varpi^t)$, and therefore

$$\mathrm{inv}_{\mathcal{P}} h(x) = \begin{cases} h(HH) & \text{if } x = HH \\ h(TT) & \text{if } x = TT \\ \dfrac{h(HT) + h(TH)}{2} & \text{if } x = HT \text{ or } x = TH. \end{cases}$$

All permutation invariant expectation operators have the following form:

$$E_q(h) = h(HH)q([HH]) + \frac{h(HT) + h(TH)}{2} q([HT]) + h(TT)q([TT]),$$

where q is any mass function on $\mathcal{A}_{\mathcal{P}}$. All strongly permutation invariant lower previsions have the following form

$$\begin{aligned} \underline{P}(h) &= \min\{E_q \colon q \in \mathcal{M}\} \\ &= \min\left\{ h(HH)q([HH]) + \frac{h(HT) + h(TH)}{2} q([HT]) + h(TT)q([TT]) \colon q \in \mathcal{M} \right\}, \end{aligned}$$

where $\mathcal{M}$ is any credal set on $\mathcal{A}_{\mathcal{P}}$.

As a special example, the representing mass functions on $\mathcal{A}_{\mathcal{P}}$ for the independent and permutation invariant precise models are given by

$$q_r([HH]) = r^2 \text{ and } q_r([HT]) = 2r(1-r) \text{ and } q_r([TH]) = (1-r)^2,$$

for $r \in [0, 1]$. □